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# Implementation of quantum algorithms with resonant interactions

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## Abstract

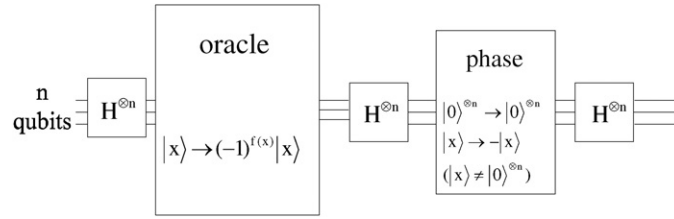
We propose a scheme for implementing quantum algorithms with resonant interactions. Our scheme only requires resonant interactions between two atoms and a cavity mode, which is simple and feasible. Moreover, the implementation would be an important step towards the fabrication of quantum computers in cavity QED system.

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Construction of quantum computers is an enormously appealing task because of quantum computational potential to perform superfast quantum algorithms. Two classes of quantum logarithms illustrate the great theoretical promise of quantum computers. One is based on Shor's Fourier transformation including quantum factoring [1], Deutsch–Jozsa logarithm [2] and so on, which are all exponential speedup compared with the classical algorithms. The other is based on Grover's quantum search [3], which is quadratic speedup compared with the classical ones. The Grover search algorithm is very important because many techniques based on the search algorithm are universally used in our lives. The Grover search algorithm is efficient to look for one item in an unsorted database of size  $N \equiv 2^n$  [3, 4]. Classically, in order to achieve the task, one needs  $O(N)$  queries. However, one needs  $O(\sqrt{N})$  queries by the Grover search algorithm. Furthermore, the efficiency of the algorithm has been manipulated experimentally in few-qubit cases via nuclear magnetic resonance (NMR) [5, 6] and optics [7, 8].

Here, we first review the general Grover search algorithm. The circuit diagram of the Grover search algorithm with  $n$  data qubits and one auxiliary working qubit, which can be used to search one item from  $2^n$  items, is shown in figure 1. The process can be concluded as the following seven steps.

- (i) Prepare the  $n + 1$  qubits, which are in  $|0\rangle^{\otimes n} |1\rangle_{n+1}$ .
- (ii) Perform the  $n + 1$  Hadamard transformations on the  $n + 1$  qubits.
- (iii) Apply the oracle. The auxiliary working qubit can be omitted after the step.



**Figure 1.** The circuit diagram for the Grover search algorithm.  $H^{\otimes n}$  denotes  $n$  Hadamard transformations on the  $n$  data qubits. An auxiliary working has been omitted. If  $x$  is the one to be searched,  $f(x) = 1$ , otherwise,  $f(x) = 0$ .

- (iv) Perform the  $n$  Hadamard transformations on the  $n$  data qubits.
- (v) Apply a phase shift to the data qubits except  $|0\rangle^{\otimes n}$ , which can be described by the unitary operator  $2|0\rangle^{\otimes n}\langle 0| - I$ , where  $I$  is the identity operation on the data qubits.
- (vi) Perform the  $n$  Hadamard transformations on the  $n$  data qubits again.
- (vii) Repeat steps 3  $\rightarrow$  6 with a finite number of times, then measure the  $n$  data qubits.

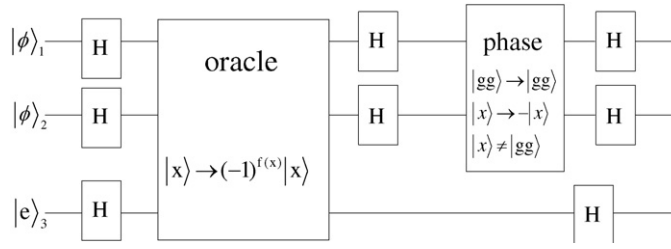
The number of repetitions [9] for obtaining a finite item is  $R = CI\left(\frac{\arccos\sqrt{1/N}}{2\arccos\sqrt{N-1/N}}\right)$ , which is bounded above by  $\pi\sqrt{N}/4$ .

On the other hand, in the realm of atom, cavity QED techniques, where atoms interact with a quantized electromagnetic field, have been proved to be a promising candidate for realizing the quantum processors. Recently, many schemes of quantum algorithms have been proposed based on cavity QED techniques. For example, Rauschenbeutel *et al* [10] have realized a two-qubit phase gate experimentally with resonant interaction of a two-level atom with a cavity mode and Zheng [11] has realized a two-qubit controlled-phase gate with resonant interaction of two three-level atoms with a cavity mode. The Deutsch–Jozsa (D–J) logarithm [12] and the Grover search algorithm [13, 14] have been realized in cavity QED, and so on.

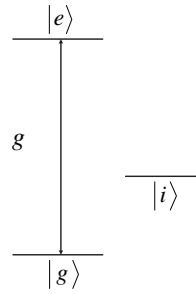
In this paper, we first mainly propose a simple scheme for implementing the Grover search algorithm in cavity QED. Comparing [13, 14], they are both based on non-resonant interactions, our scheme is based on single resonant interactions between atoms and cavity and does not use the cavity mode as the data bus. Thus the current scheme is very simple and the interaction time is very short, which is important in view of decoherence. More importantly, we strictly investigate the case of atomic spontaneous emission and cavity decay during the interactions. Then we avoid the effect by constructing appropriate unitary transformations. Therefore our proposal is more approach to real case and can succeed with higher fidelity (over 0.99). Here, we only discuss the case of two data qubits, where we can search a finite item from four items. The circuit diagram of the Grover search with two data qubits and one auxiliary working qubit is shown in figure 2. Three-level atoms are used in this paper and the relevant level structure is shown in figure 3. The third level  $|i\rangle$  is not affected during the atom–cavity resonant interaction. Thus we consider the case that two atoms interact with the single cavity mode, in the interaction picture, the Hamiltonian of the atom–cavity interaction can be expressed as (assuming  $\hbar = 1$ ) [11]

$$H = g_1(a^\dagger S_1^- + a S_1^+) + g_2(a^\dagger S_2^- + a S_2^+), \quad (1)$$

where  $g_1$  and  $g_2$  are the coupling strengths of the atoms 1, 2 with the cavity, respectively.  $s^+ = |e\rangle\langle g|$ ,  $s^- = |g\rangle\langle e|$  and  $|g\rangle$  is the ground state of the atoms,  $|e\rangle$  is the excited state of the atoms.  $a^\dagger, a$  are the creation and annihilation operators of the cavity mode. Assume



**Figure 2.** The circuit diagram for the two-data-qubit Grover search algorithm.  $H$  denotes Hadamard transformation.  $|\phi\rangle_1$  and  $|\phi\rangle_2$  are two data qubits,  $|e\rangle_3$  is an auxiliary working qubit.



**Figure 3.** The level structure of the atoms.  $|g\rangle$  is the ground state,  $|e\rangle$  is the excited state. The cavity mode is resonantly coupled to the  $|e\rangle \leftrightarrow |g\rangle$  transition. The third level  $|i\rangle$  is not affected by the interaction.

that the cavity mode is initially prepared in the vacuum state  $|0\rangle_c$ . In order to implement the two-data-qubit Grover search algorithm, firstly, we prepare atoms 1, 2 and 3 in the state

$$|\phi\rangle_{123} = |gge\rangle_{123} \tag{2}$$

and send atoms 1 and 3 through a classical field and choose appropriately phase and amplitude, respectively

$$|g\rangle_1 \rightarrow \frac{1}{\sqrt{2}}(|g\rangle_1 + |e\rangle_1), \tag{3a}$$

$$|e\rangle_3 \rightarrow \frac{1}{\sqrt{2}}(|g\rangle_3 - |e\rangle_3). \tag{3b}$$

Then we send atom 2 through two classical fields and choose appropriately phases and amplitudes

$$|g\rangle_2 \rightarrow \frac{1}{\sqrt{2}}(|g\rangle_2 + |e\rangle_2) \rightarrow \frac{1}{\sqrt{2}}(|g\rangle_2 + |i\rangle_2). \tag{4}$$

So the total state of the atoms 1, 2 and 3 becomes

$$|\phi\rangle_{123} = \frac{1}{2\sqrt{2}}(|gg\rangle_{12} + |gi\rangle_{12} + |eg\rangle_{12} + |ei\rangle_{12})(|g\rangle_3 - |e\rangle_3). \tag{5}$$

Obviously, we know that the four items  $(|gg\rangle_{12}, |gi\rangle_{12}, |eg\rangle_{12}, |ei\rangle_{12})$  are stored in the data qubits before applying the oracle. Without loss of generality, we search the state  $|eg\rangle_{12}$  from the four states. However, for the two-data-qubit Grover search algorithm, the oracle has effect on the states to be searched. The auxiliary working qubit can be discarded at this point.

Secondly, we send atoms 1 and 2 through the vacuum cavity, the evolutions are governed by the Hamiltonian of equation (1),

$$|eg\rangle_{12}|0\rangle_c \rightarrow \frac{g_1}{E} \left\{ \frac{1}{E} \left( g_1 \cos(Et) + \frac{g_2^2}{g_1} \right) |eg\rangle_{12}|0\rangle_c + \frac{1}{E} g_2 [\cos(Et) - 1] |ge\rangle_{12}|0\rangle_c - i \sin(Et) |gg\rangle_{12}|1\rangle_c \right\}, \quad (6a)$$

$$|ei\rangle_{12}|0\rangle_c \rightarrow [\cos(g_1 t) |e\rangle_1 |0\rangle_c - i \sin(g_1 t) |g\rangle_1 |1\rangle_c] |i\rangle_2, \quad (6b)$$

$$|gg\rangle_{12}|0\rangle_c \rightarrow |gg\rangle_{12}|0\rangle_c, \quad (6c)$$

$$|gi\rangle_{12}|0\rangle_c \rightarrow |gi\rangle_{12}|0\rangle_c, \quad (6d)$$

where  $E = \sqrt{g_1^2 + g_2^2}$ . If we choose

$$t = \frac{\pi}{g_1}, \quad g_2 = \sqrt{3}g_1, \quad (7)$$

which can be achieved by choosing coupling strengths and interaction time appropriately. Thus, we have

$$|eg\rangle_{12}|0\rangle_c \rightarrow |eg\rangle_{12}|0\rangle_c, \quad (8a)$$

$$|ei\rangle_{12}|0\rangle_c \rightarrow -|ei\rangle_{12}|0\rangle_c, \quad (8b)$$

$$|gg\rangle_{12}|0\rangle_c \rightarrow |gg\rangle_{12}|0\rangle_c, \quad (8c)$$

$$|gi\rangle_{12}|0\rangle_c \rightarrow |gi\rangle_{12}|0\rangle_c. \quad (8d)$$

Then send atom 2 through two classical fields tuned to the transition

$$|i\rangle_2 \rightarrow |e\rangle_2, \quad |g\rangle_2 \longleftrightarrow |e\rangle_2. \quad (9)$$

These lead the state of atoms 1 and 2 to

$$|\phi\rangle_{12} = \frac{1}{2}(|gg\rangle_{12} + |ge\rangle_{12} - |eg\rangle_{12} + |ee\rangle_{12}). \quad (10)$$

Thirdly, we send atoms 1 and 2 through a classical field, respectively. Choosing appropriately phase and amplitude, let

$$|g\rangle_i \rightarrow \frac{1}{\sqrt{2}}(|g\rangle_i + |e\rangle_i), \quad (i = 1, 2) \quad (11a)$$

$$|e\rangle_i \rightarrow \frac{1}{\sqrt{2}}(|g\rangle_i - |e\rangle_i), \quad (i = 1, 2). \quad (11b)$$

Then we perform a single-qubit operation on atom 2 again

$$|e\rangle_2 \rightarrow |i\rangle_2. \quad (12)$$

Thus equation (10) becomes

$$|\phi\rangle_{12} = \frac{1}{2}(|gg\rangle_{12} + |eg\rangle_{12} - |gi\rangle_{12} + |ei\rangle_{12}). \quad (13)$$

In order to achieve the next step (phase transformation), we can perform single-qubit operations and controlled-phase transformations on the two atoms as in equation (8), which can lead equation (13) to

$$|\phi\rangle_{12} = \frac{1}{2}(|gg\rangle_{12} + |ge\rangle_{12} - |eg\rangle_{12} - |ee\rangle_{12}). \quad (14)$$

Finally, we perform single-qubit operations on atoms 1 and 2 as in equation (11). Thus we obtain the state of atoms 1 and 2

$$|\phi\rangle_{12} = |eg\rangle_{12}. \tag{15}$$

We can measure the state of atoms 1 and 2 by detectors. Obviously, the state of atoms 1 and 2 is the result that we want to search. If we want to search other states ( $|gg\rangle_{12}$ ,  $|gi\rangle_{12}$  or  $|ei\rangle_{12}$ ), the main process is the same as above (equations (8) and (11)), except for some single-qubit operations.

But in the real processing of resonant interactions, the cavity decay and the atomic spontaneous emission are unavoidable. Taking them into consideration, if we choose appropriate parameters  $g_2 = \sqrt{3}g_1$ ,  $t = \frac{\pi}{g_1}$  and  $\kappa = \tau = 0.1g_1$ , where  $\kappa$  is the cavity decay rate and  $\tau$  is the atomic spontaneous emission rate, the evolution [11] of system is similar to equation (8). While  $10^{-\pi/20}$  is added to the  $|eg\rangle_{12}$  and  $-|ei\rangle_{12}$  compared with the ideal case. The state of equation (14) becomes

$$|\phi\rangle_{12} = \frac{1}{\sqrt{1 + 2 \times 10^{-\frac{\pi}{10}} + 10^{-\frac{\pi}{5}}}}(|gg\rangle_{12} - 10^{-\frac{\pi}{20}}|ge\rangle_{12} + 10^{-\frac{\pi}{20}}|eg\rangle_{12} - 10^{-\frac{\pi}{10}}|ee\rangle_{12}). \tag{16}$$

Then we perform the single-qubit operations

$$|g\rangle_1 \rightarrow \frac{1}{\sqrt{1 + 10^{-\pi/10}}}(10^{-\pi/20}|g\rangle_1 + |e\rangle_1), \tag{17a}$$

$$|e\rangle_1 \rightarrow \frac{1}{\sqrt{1 + 10^{-\pi/10}}}(10^{-\pi/20}|g\rangle_1 - |e\rangle_1), \tag{17b}$$

and

$$|g\rangle_2 \rightarrow \frac{1}{\sqrt{1 + 10^{-\pi/10}}}(10^{-\pi/20}|g\rangle_2 + |e\rangle_2), \tag{18a}$$

$$|e\rangle_2 \rightarrow \frac{1}{\sqrt{1 + 10^{-\pi/10}}}(10^{-\pi/20}|g\rangle_2 - |e\rangle_2), \tag{18b}$$

on atoms 1 and 2, respectively. These lead the state of atoms 1 and 2 to  $|eg\rangle_{12}$ , i.e., we can search the state perfectly (the successful possibility and fidelity are both equal to 1.0).

Out of question, the D–J algorithm can also be implemented with resonant interactions. The D–J algorithm can distinguish the function  $f(x)$  between constant and balanced [2]. The values of the function  $f(x)$  are either 0 or 1 for all possible inputs. The values of balance function are equal to 1 for half of all the possible inputs, and 0 for the other half. The constant is always 1 or 0 for all inputs. Classically, if we want to unambiguously distinguish between constant and balanced function on  $2^n$  inputs, we will need  $2^n/2 + 1$  queries to achieve the task. While for the D–J algorithm, we will need only one query. Here we discuss the two-qubit D–J algorithm. The state of query and auxiliary working qubit is prepared in  $(|0\rangle_i + |1\rangle_i)(|0\rangle_a - |1\rangle_a)/2$ . After mapping a unitary transformation  $U_f$  on the system, the initial state becomes  $[(-1)^{f(0)}|0\rangle_i + (-1)^{f(1)}|1\rangle_i](|0\rangle_a - |1\rangle_a)/2$ . There are four possible transformations to the  $U_f$ : (1) for  $U_{f1}$ ,  $f(0) = f(1) = 0$ ; (2) for  $U_{f2}$ ,  $f(0) = f(1) = 1$ ; (3) for  $U_{f3}$ ,  $f(0) = 0$  and  $f(1) = 1$ ; (4) for  $U_{f4}$ ,  $f(0) = 1$  and  $f(1) = 0$ . After a Hadamard transformation on the query qubit, the state of query qubit becomes  $|f(0) \oplus f(1)\rangle$ . If the function  $f(x)$  is constant, the state of query qubit becomes  $|0\rangle_i$ . Otherwise it becomes  $|1\rangle_i$ .

Obviously implementation of the unitary transformation  $U_f$  is the key. We prepare two atoms in

$$|\varphi\rangle_{12} = (|g\rangle_1 + |e\rangle_1)(|g\rangle_2 - |e\rangle_2)/2. \quad (19)$$

In the case of  $U_{f1}$ , we take no operation on the two atoms.

In the case of  $U_{f2}$ , we perform a single-qubit rotation on atom 2,

$$|g\rangle_2 \longleftrightarrow |e\rangle_2. \quad (20)$$

In the case of  $U_{f3}$ , we first perform a single-qubit rotation on atom 2,

$$|e\rangle_2 \longleftrightarrow |i\rangle_2. \quad (21)$$

Secondly we send atoms 1 and 2 through a vacuum cavity. We can obtain the evolution of equation (8), which is governed by the Hamiltonian of equation (1). Thirdly we perform single-qubit rotation on atom 2 of equation (21) and another single-qubit rotation of equation (20). Then we send atoms 1 and 2 through the vacuum cavity again as equation (8). Finally, we perform single-qubit rotation on atom 2 of equation (21).

In the case of  $U_{f4}$ , we can achieve the task by the process of the case of  $U_{f3}$  and a single-qubit operation on atom 2.

Now, we have completed the unitary  $U_f$ . Thus the two-qubit D–J algorithm will be implemented simply. Moreover, the scheme can be generalized to multi-qubit case.

For the real processing (with cavity decay and the atomic spontaneous emission), we can also achieve the task with successful possibility and fidelity being both 1.0 by choosing appropriate single-qubit operations for the implementation of Grover search algorithm. This is not only a useful character in experimental manipulation but also important for constructing real quantum computer.

Discussion on the feasibility of the current scheme is necessary. The scheme requires two atoms in a vacuum cavity having different coupling strengths with the cavity mode. The coupling depends on the atomic positions:  $g = \Omega e^{-r^2/\omega^2}$ , where  $\Omega$  is the coupling strength at the cavity centre,  $\omega$  is the waist of the cavity mode, and  $r$  is the distance between the atom and the cavity centre [15]. The condition  $g_2 = \sqrt{3}g_1$  in our scheme can be satisfied by locating one atom at the centre of the cavity and locating the other one at the position  $r = \omega \ln^{1/2} \sqrt{3}$ . According to the recent experiments with Cs atoms trapped in an optical cavity [16], the condition can be obtained. For the resonant cavity, in order to implement quantum algorithms successfully, the relationship between the interaction time and the excited atom lifetime should be taken into consideration. The interaction time should be much shorter than that of atom radiation. Hence, atom with a sufficiently long excited lifetime should be chosen. For Rydberg atoms with principal quantum numbers 50 and 51, the radiative time is  $T_1 \simeq 3 \times 10^{-2}$  s. From the analysis in [17], the interaction time is on the order  $T \simeq 2 \times 10^{-4}$  s, which is much shorter than the atomic radiative time. So the condition can be satisfied by choosing Rydberg atoms. Furthermore our scheme requires that two atoms be simultaneously sent through a cavity, otherwise there will be an error. Assume that during the interaction between the two atoms and the cavity, one atom enters the cavity  $0.01t$  sooner than another atom, with  $t$  being the time of each atom staying in the cavity. We can obtain the fidelity  $F \simeq 0.999$  for equation (10) and the total fidelity is about 0.998 for the two data qubits Grover algorithm. Obviously in this case the operation is only slightly affected in current schemes.

Next, one needs to reach the Lamb–Dicke regime in order to implement the quantum algorithms. For the state of equation (5), in the Lamb–Dicke regime, the infidelity caused by the spatial extension of the atomic wavefunction is about  $\Delta \simeq (ka)^2\pi$ , where  $k$  is the wave vector of the cavity mode and  $a$  is the spread of the atomic wavefunction. Setting  $\Delta \simeq 0.01$ ,

so we have  $a \simeq 0.01\lambda$ , where  $\lambda$  is the wavelength of the cavity mode. If the atom trajectories cross the cavity with the deviation of less than 0.1 degree from its pre-determined direction, we can ensure the fidelity is about 0.999 for equation (10) and the unitary transformation  $U_f$ . While in order to maintain  $g_2 = \sqrt{3}g_1$  in the process of atomic motion in the cavity, we can choose the parameter of cavity  $z \leq 0.5z_0$ , where  $z_0 = \frac{\pi\omega^2}{\lambda}$  and  $2z$  is the length of the cavity. We can obtain that the error is only about  $10^{-3}$ . Therefore our scheme is feasible with the current cavity QED technology.

In conclusion, we have proposed the scheme for implementing the quantum algorithms in cavity QED. Our scheme only requires resonant interactions between two atoms and a cavity mode. The interaction time is very short, which is very important in view of decoherence. Meanwhile, even if we take the cavity decay and atomic spontaneous emission into consideration, we can still achieve the task perfectly. Moreover, the implementation of the algorithms would be an important step to scale more complex quantum algorithms and our scheme would be very important for constructing real quantum computer.

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